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SMALL CANCELLATION THEORY WITH A WEAKENED SMALL CANCELLATION HYPOTHESIS. II. THE WORD PROBLEM

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ABSTRACR
In [1] we introduced the small concellation condition W(4) and developed the basic theory for groups having a W(4) presentation. In this paper we solve the word problem for groups with a finite W(4) presentation.

Introduction

In [1] we introduced the geometrical small cancellation condition W(4) and developed the basic theory. In this work we solve the word problem for groups having a finite presentation satisfying the condition W(4). In fact, we prove the following Area Theorem:

THEOREM A. *Let M be a simply connected map which contains more than one region and has connected interior. Denote by $\beta(M)$ the number of boundary regions of M which contain an edge on the boundary of M and let $V(M)$ be the number of regions of M. If M satisfies* W(4) *then* $V(M) \leqq \beta(M)^2$.

The solution of the word problem follows easily from the Area Theorem (see [2, p. 262]).

The proof of the Theorem is by induction on $V(M)$. We show that we can always delete a part of the boundary layer of M (see the definition below) such that the remaining map M' is simply connected with connected interior and

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$\beta(M') \leqq \beta(M)$ (Theorem 1). Now the theorem follows by an easy calculation (Remark 1). The main step of the proof is the construction of a set of boundary chains which diminish the length $\beta(M)$ of the boundary. These are given in Definition 3.

For notation and unexplained terms see [1] and [2].

Proof of the Area Theorem

We shall prove the Area Theorem by induction on $V(M)$. The induction step is based on the following easy remark.

REMARK 1. Let M be a connected and simply connected map. Assume that M has a submap S with s regions such that if M' is the submap obtained by deleting S from M then

(a) M' is connected and simply connected;

(b) $\beta(M') \leqq \beta(M) - 1$;

(c) $s \leqq \beta(M)$.

Suppose $V(M') \leqq \beta(M')^2$. Then $V(M) \leqq \beta(M)^2$.

Indeed,

$$\begin{aligned} V(M) &\leqq V(M') + \beta(M) \\ &\leqq \beta(M')^2 + \beta(M) \\ &\leqq (\beta(M) - 1)^2 + \beta(M) \\ &\leqq \beta(M)^2. \end{aligned}$$

In the rest of the paper we show that we can always find a submap S as in the Remark. We describe these submaps in Definition 3. But first we need two further notions defined below. (See Definitions 1 and 2.)

DEFINITION 1 (Boundary strips and special strips).

(a) (See [1, Def. 5.6(c)].) Let M be a connected and simply connected map. Let S be a connected submap of M consisting of regions $D_1, \ldots, D_r$ and let M' be the map obtained from M by deleting all the regions of S.

S is called a *boundary strip in M* if the following hold:

(i) M' is connected (hence nonempy);

(ii) S is either simply connected or annular;

(iii) $\partial D_j \cap \partial M$ is connected and contains an edge $j = 1, \ldots, r$;

(iv) $\partial D_i \cap \partial M' \neq \emptyset$, $j = 1, \ldots, r$;

(v) $\partial D_j \cap D_{j+1}$ contains an edge, $j = 1, \ldots, r-1$.

(b) Let M be a connected and simply connected map and let S be a boundary strip of M. For $D \in S$, let M_D be the map obtained by deleting D from M. S is a *special strip* if the following holds: If $D' \in E$ is adjacent to D then $D' \in \operatorname{Cor}(M_D)$. (See [1, Def. 5.6].) See Fig. 1.

DEFINITION 2 (Boundary chains). Let M be a simply connected map with a connected interior and let C be a submap of M. Let M' be the submap of M obtained by removing C from M. C is a *boundary chain of* M if the following hold:

(1) There are special strips $C_1, \ldots, C_t$ in M such that $C = \bigcup_{i=1}^{t} C_i$. We call C_i the *components* of C.

(2) If $\partial C_i \cap \partial C_j$ is not empty then $\partial C_i \cap \partial C_j$ consists of a single (boundary) vertex except for the case when C is annular and $t = 2$. In this case $\partial C_1 \cap \partial C_2$ consists of two vertices. See Fig. 2(b).

(3) $\partial C_j \cap \partial C_{j+1} \neq \varnothing$ for $j = 1, \ldots, t-1$. (See Fig. 2.)

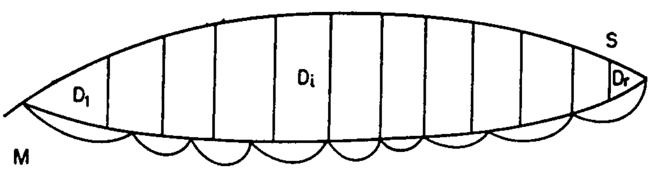
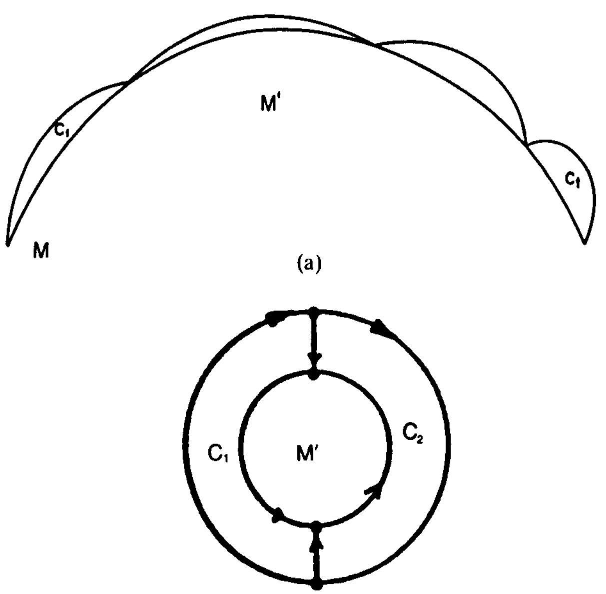


Fig. 1.



(a)

(b)

Fig. 2.

A boundary chain C is *maximal in* M if no boundary chain of M contains C properly.

DEFINITION 3 (Reducing chains). Let M be a simply connected map with a connected interior and let S be a submap of M.

(a) S is a *reducing chain of the first kind* if S consists of a single region D, $D \in \mathrm{Cor}_1(M)$ and at least one of the endpoints of $\partial D \cap \partial M$ has valency 3 in M (see Fig. 3(a)). For the definition of $\mathrm{Cor}_k(M)$ see [1, p. 85].

(b) S is a *reducing chain of the second kind* if S consists of a single region D, $D \in \mathrm{Cor}_2(M)$ such that both endpoints of $\partial D \cap \partial M$ have valency 3 in M. (See Fig. 3(b).)

(c) S is a *reducing chain of the third kind* if S consists of two regions D_1 and D_2 in $\mathrm{Cor}_2(M)$ such that the endpoints of $\partial D_1 \cap \partial D_2$ have valency 3 in M. (See Fig. 3(c).)

(d) S is a *reducing chain of the fourth kind* if S consists of two regions D_1 and D_2, $D_1 \in \mathrm{Cor}_2(M)$ and $D_2 \in \mathrm{Cor}_3(M)$ such that the endpoints of $\partial D_2 \cap \partial M$ and $\partial D_2 \cap \partial D_1$ have valency 3 in M. (See Fig. 3(d).)

(e) S is a *reducing chain of the fifth kind* if S consists of two regions D_1 and D_2, D_1, $D_2 \in \mathrm{Cor}_3(M)$ such that $\partial D_1 \cap \partial D_2$ contains an edge and the endpoints of $\partial D_i \cap \partial M$, $i = 1,2$ and of $\partial D_1 \cap \partial D_2$ have valency 3 in M. (See Fig. 3(e).)

(f) S is a *reducing chain of the sixth kind* if S is a boundary strip consisting of three regions D_1, D_2 and D_3 such that $D_1, D_3 \in \mathrm{Cor}_2(M)$, $i_M(D_2) = 4$ and there are regions E_1 and E_2 of M such that E_1 is a common neighbour of D_1 and D_2 and E_2 is a common neighbour of D_2 and D_3. (See Fig. 3(f).)

(g) S is an *exceptional reducing chain* if the following hold:

(i) S is a simply connected boundary chain with components $C_1, \dots, C_r$, $r \geqq 2$.

(ii) Every component C_i consists of a single region D_i.

(iii) The components of S can be labelled in such a way that for $i = 1, \dots, r-1$ D_i and D_{i+1} have a common neighbour in M and $D_j \in \mathrm{Cor}_2(M)$ for $j = 2, \dots, r-1$, while D_1, $D_r \in \mathrm{Cor}_2(M) \cup \mathrm{Cor}_1(M)$.

(iv) If $D_j \in \mathrm{Cor}_2(M)$ for $j = 1$ or $j = r$ (or both) then the endpoint of $\partial S \cap \partial M$ which is contained in ∂D_j has valency 3 in M. (See Fig. 3(g).)

REMARK 2. Let M be a simply connected map with a connected interior and let S be a reducing chain of M. Let M' be the submap of M obtained by deleting S from M. The $\beta(M') < \beta(M)$.

In view of Remarks 1 and 2, Theorem A will follow from the following theorem.

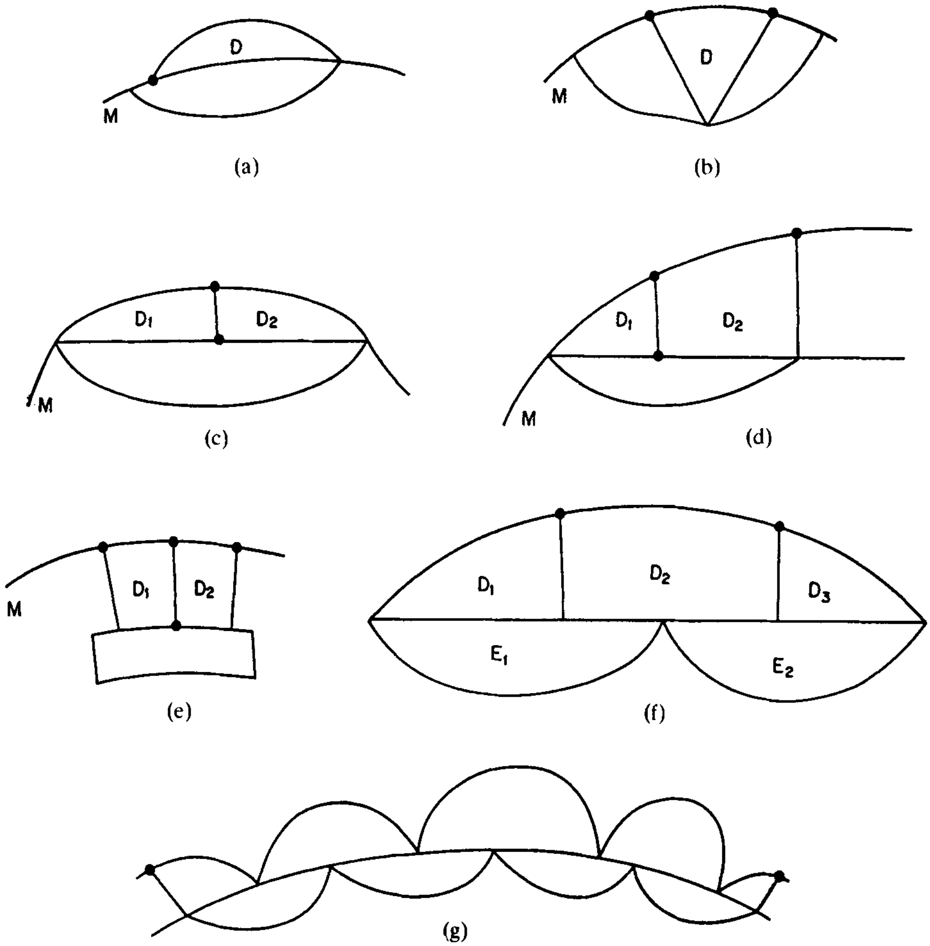


Fig. 3.

THEOREM 1. *Let M be a simply connected map with a connected interior. Assume that M satisfies the condition* W(4). *If M contains more that one region and M has no boundary regions with one neighbour, then M has a reducing chain.*

The way we shall prove the theorem is to deduce a contradiction from the existence of a minimal counterexample, through Lemmas 1–4. To this end we introduce the following hypotheses.

$\mathcal{K}_1$: M satisfies the conditions of Theorem 1.

$\mathcal{K}_2$: (a) Every proper simply connected submap of M with more than one region and with a connected interior has a reducing boundary chain;

(b) M doesn't have a reducing boundary chain.

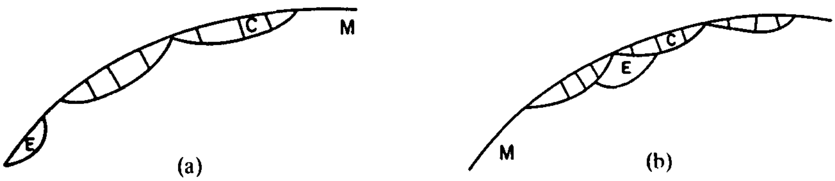


Fig. 4.

Lemma 1. *Let M be a simply connected map with a connected interior. Let C be a maximal boundary chain of M and let M' be the submap of M obtained by deleting C from M. Let M'' be a connected component of M'. Let $E \in \mathrm{Cor}(M'')$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$. Then one of the following holds* (see Fig. 4).

(a) $\partial E \cap \partial C = \varnothing$ (Fig. 4(a)).

(b) $\partial E \cap \partial M'' \subseteq \partial C$ (Fig. 4(b)).

Proof. Assume that both (a) and (b) fail to hold. Then $\partial E \cap \partial M$ contains an edge and C is not annular. Let us consider $\partial E \cap \partial C$. Since $E \in \mathrm{Cor}(M'')$, either $\partial E \cap \partial C$ is connected or the path τ describing $\partial E \cap \partial M''$ has a decomposition $\tau = \tau_1\tau_2\tau_3$ such that $\partial E \cap \partial C = \tau_1 \cup \tau_3$ (see Fig. 5(a)). Assume that $\partial E \cap \partial C$

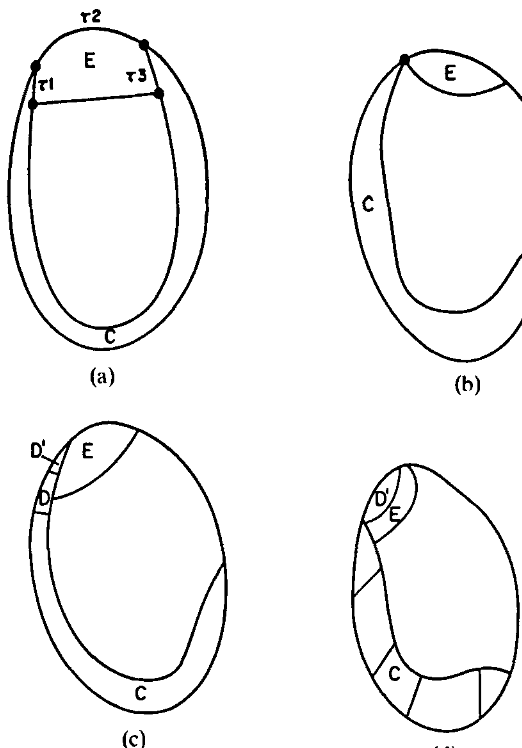


Fig. 5.

does not contain an edge. Then $\partial E \cap \partial C$ consists of one or two single vertices. Assume first that $\partial E \cap \partial C$ consists of a single vertex v (see Fig. 5(b)). Since $E \in \mathrm{Cor}(M'')$, we may adjunct E to C and get a boundary chain $C_1 = C \cup \{E\}$ which properly contains C, contradicting the maximality of C. Similarly for two vertices. So let μ describe a component of $\partial E \cap \partial C$. Let D' and D'' be the extremal regions of C. Then at least one of them, say D', contains a head (tail) μ_0 of μ on its boundary (see Fig. 5(c)). We claim that $\mu = \mu_0$. For if not then $\{D'\}$ is a reducing chain of the second kind if $\partial D' \cap \partial D$ contains an edge for some D in C (see Fig. 5(c)) and $\{D'\}$ is a reducing chain of the first kind, if $\partial D' \cap \partial D$ consists of a single vertex, for some D in C. (See Fig. 5(d).)

However, this violates $\mathcal{H}_1$ or $\mathcal{H}_2$ in both cases. Thus $\partial E \cap \partial D'$ is a component of $\partial E \cap \partial C$. Similarly, if $\partial E \cap \partial D'' \neq \emptyset$ then $\partial E \cap \partial D''$ is a component of $\partial E \cap \partial C$. But then $C \cup \{E\}$ is a boundary chain of M which properly contains C, contradicting the maximality of C in M. Consequently, $\partial E \cap \partial C = \emptyset$, contradicting that both (a) and (b) fail to hold.

Lemma 2. *Let M be a simply connected map with a connected interior and assume that C is a maximal boundary chain of M. Let M' be the submap of M obtained by deleting C from M and let S be a reducing chain of M'. If $\mathcal{H}_1$ and $\mathcal{H}_2$ hold then*

(a) *S is not of the sixth kind and*

(b) *$\partial S \cap \partial M' \subseteq \partial C$.*

Proof. Note that by Definition 3 every region of a reducing chain belongs to $\mathrm{Cor}(M')$, except for the case when S is of the sixth kind. Hence (a) implies (b) by Lemma 1. So we prove (a). Assume S if of the sixth kind and let $S = \{D_1, D_2, D_3\}$ as defined by Definition 3(f). (See Fig. 3(f).) Let $\mu = \partial S \cap \partial M'$. Since $D_1, D_3 \in \mathrm{Cor}_2(M)$, we have three cases to consider, due to Lemma 1.

Case 1. $\mu \subseteq \partial D_2$ (see Fig. 6(a)).

Case 2. $\mu \cap \partial D_1 \subseteq \mu$ (see Fig. 6(b)).

Case 3. $\partial D_3 \cap \partial M \subseteq \mu$ (see Fig. 6(c)).

Case 1: Since $\mu \cap \partial D_i = \emptyset$ for $i = 1, 3$, μ cannot have an endpoint with valency greater than 3 because then M would have a reducing chain of the second kind, containing this vertex. So we may assume that the endpoints of μ have valency 3. But then it easily follows that C contains a reducing chain of at least one of the kinds 1, 3, 4 described in Definition 3. This contradicts $\mathcal{H}_1$ or $\mathcal{H}_2$.

Case 2: Since D_1 has an inner vertex with valency 3 in M, the W(4) condition implies that $i_M(D_1) \geqq 5$. But since $i_{M'}(D_1) = 2$, D_1 has at least three

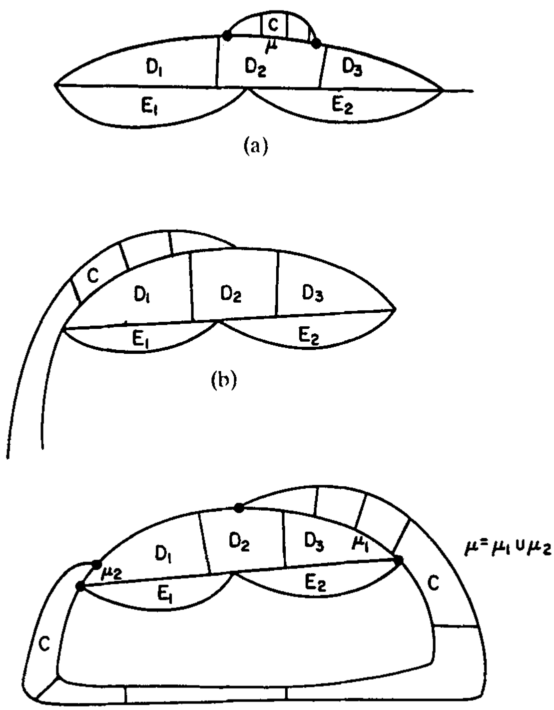


Fig. 6.

neighbours in C, hence at least 2 vertices on μ which are not endpoints of μ. If $i_M(D_1)=5$ then these vertices have valency at least 4, hence M has a reducing chain of the second kind. Thus we may assume that these vertices have valency 3 and consequently $i_M(D_1)\geqq 6$, by the W(4) condition. But then C contains a reducing subchain of the fifth kind, again contradicting $\mathcal{K}_1$ or $\mathcal{K}_2$.

Case 3: Similar to Case 2.

LEMMA 3. *Let M be a simply connected map with a connected interior. Let C be a boundary chain of M and let M' be the submap obtained by deleting C from M. Let $E \in \mathrm{Cor}(M')$ and assume that $\partial E \cap \partial M' \subseteq \partial C$. Suppose that*

(*) *$\partial E \cap \partial M'$ has an endpoint with valency 3 in M' if $E \in \mathrm{Cor}_2(M')$ and*

(**) *$\partial E \cap \partial M'$ has both endpoints with valency 3 in M', if $E \in \mathrm{Cor}_3(M')$.*

Then one of the following holds.

(a) *M has a reducing chain; or*

(b) *M has a boundary region with one neighbour; or*

(c) $d_M(E)=4$, $E \in \mathrm{Cor}_2(M')$ *and there are extremal regions* D' *and* D'' *of* C *if* C *consists of a unique component and consecutive components of* C *which contain the endpoints* u *and* v *of* $\partial E \cap \partial C$ *respectively, if* C *contains more than one component, and*

(i) $\partial E \cap \partial C = (\partial E \cap \partial D') \cup (\partial E \cap \partial D'')$;

(ii) *either* $\partial D' \cap \partial M' = \partial D' \cap \partial E$ *in which case* $d_{M'}(u)=3$ (see Fig. 7) *or* $\partial D'' \cap \partial M = \partial D'' \cap \partial E$ *in which case* $d_{M'}(v)=3$.

Proof. Let u and v be the endpoints of $\partial E \cap \partial M'$. We distinguish three cases:

Case 1: u and v belong to the same component C_i of C. (See Fig. 8(a).)

Case 2: u and v belong to different components C_i and C_j of C respectively, such that $\partial C_i \cap \partial C_j = \varnothing$. (See Fig. 8(b).)

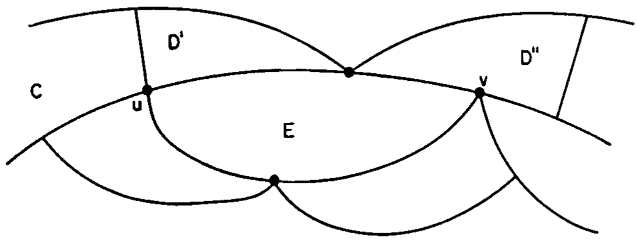


Fig. 7.

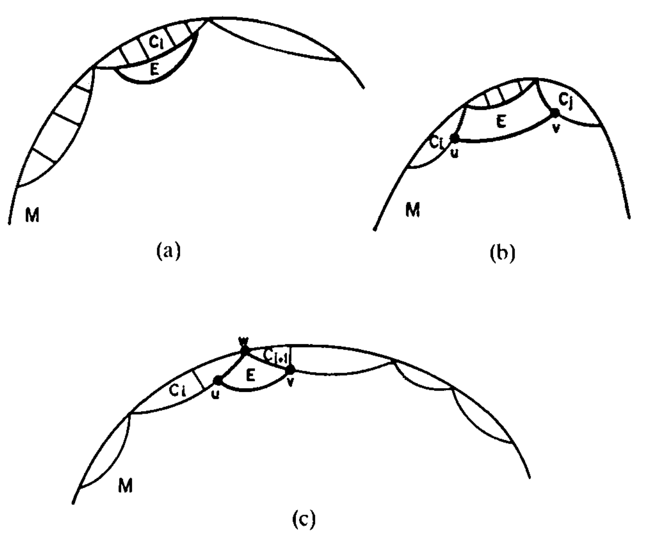


Fig. 8.

Case 3: u and v belong to different components C_i and C_j of C respectively, such that $\partial C_i \cap \partial C_j = \{w\}$ for a vertex w with valency 4 (i.e., C_i and C_j are consecutive components of C). (See Fig. 8(c).)

Case 1: Let $\mu = \partial E \cap \partial C_i$. It follows from the W(4) condition that μ has inner vertices, i.e., vertices other than u and v. Let w be such a vertex. Then we may assume that w has valency 3. For assume w has valency at least 4. If C_i is simply connected then Σ has a region D with $i(D) = 2$ such that $w \in \partial D$ and the endpoints of $\partial D \cap \partial M$ have valency 3, so M has a reducing boundary chain of the second kind. (See Fig. 9(a).) If C_i is annular and no such region D exists, then the two regions, say D_0 and D_t, which contain w on their boundary, constitute an exceptional reducing boundary chain (see Fig. 9(b)). So assume that all the inner vertices w of μ have valency 3 in M. If μ has l inner vertices and $E \in \mathrm{Cor}_k(M')$, $k = 1, 2, 3$, then $l \geqq 1$. Due to the property C $N(1)$ (see [1, 2]) and the condition W(4) we have

(1) $$d_M(E) = k + 1 + l.$$

Let us consider the cases $k = 1$, $k = 2$ and $k = 3$ separately. For the cases $k = 1,2$ we shall show $l \geqq 3$. Then C will contain a reducing chain of the fifth kind (see Fig. 9(c)). To this end, by (1) it is enough to show

(2) $$d_M(E) \geqq k + 4.$$

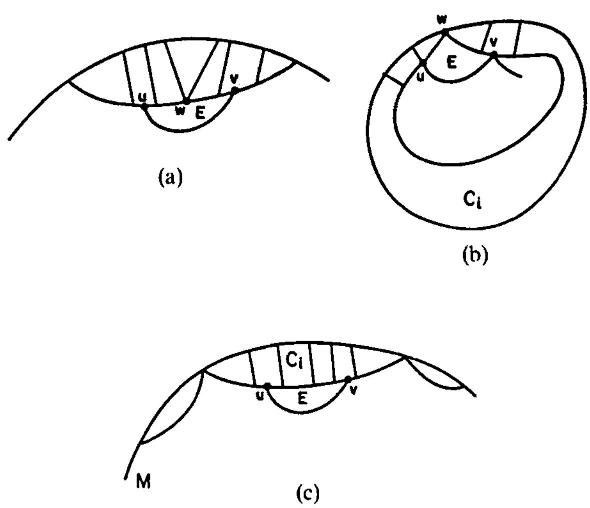


Fig. 9.

If $k = 1$ then ∂E has at most 2 vertices with valency 4, hence the condition W(4) forces $d_M(E) \geqq 7$ and (2) holds.

If $k = 2$ then ∂E has at most 3 vertices with valency 4, hence the condition W(4) forces $d_M(E) \geqq 6$ and again (2) holds.

If $k = 3$ then we distinguish two subcases.

Subcase 1: One of u or v has valency 3 in M (see Fig. 10(a)). Then ∂E has at most 2 vertices with valency 4, hence by the W(4) condition we have $d_M(E) \geqq 7$. Thus (2) holds, $l \geqq 3$ and again M has a reducing chain of the fifth kind.

Subcase 2: Both u and v have valency 4 in M (see Fig. 10 (b), (c)). Since ∂E has at most 3 vertices with valency 4, $d_M(E) \geqq 6$, hence $l \geqq 2$. Therefore M contains a reducing chain of the fourth or fifth kind. (See Fig. 10(b) and (c).)

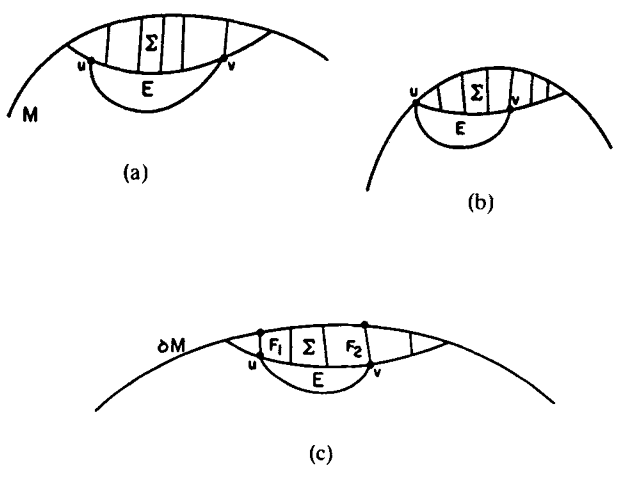


Fig. 10.

Case 2. We may assume that u and v are not on the boundary of the same component of C, in view of Case 1. Assume $j - i > 1$ and let C_k be a component of C with $i < k < j$. If C_k consists of a single region then (b) holds. (See Fig. 11(a).) So assume C_k contains at least 2 regions. If C_k contains 2 regions then C_k constitutes a reducing chain of M of the third kind, hence (a) holds. (See Fig. 11(b).) Finally, if C_k contains at least three regions, then C_k has a head consisting of two regions which constitute a reducing chain of M of the fourth kind (see Fig. 11(c)) or of the second kind (see Fig. 11(d)).

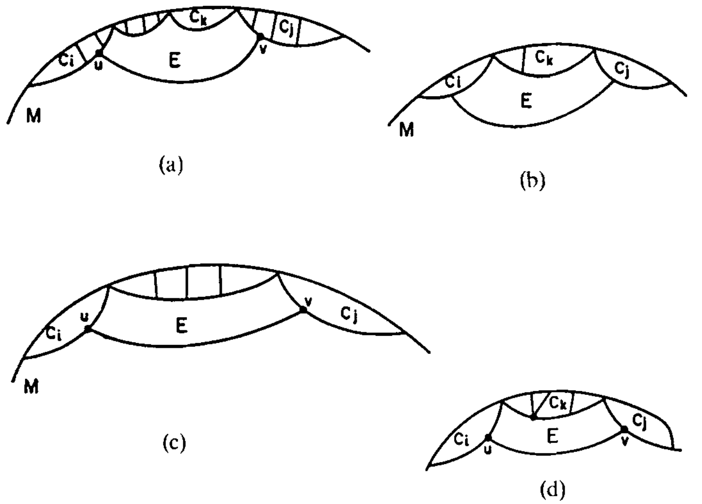


Fig. 11.

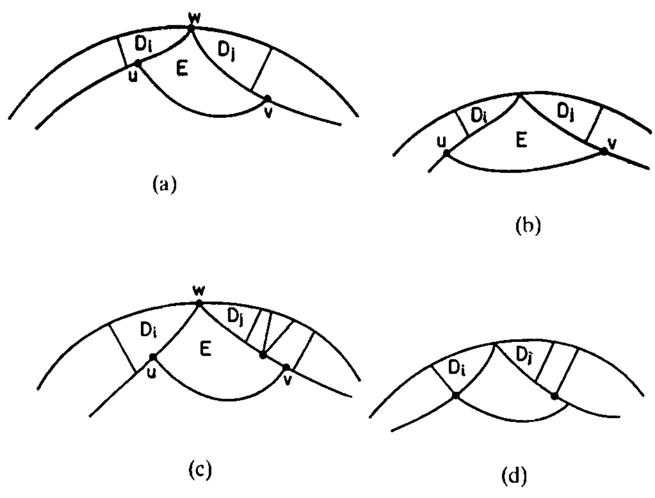


Fig. 12.

Case 3: Let D_i and D_j be the extremal regions of C_i and C_j respectively which contain w. We claim that

either $u \in D_i$ *or* $v \in D_j$ (see Fig. 12(a)).

For if not then $\{D_i, D_j\}$ constitutes an exceptional reducing chain of the second kind (see Fig. 12(b)).

Assume $u \in D_i$. Let $\mu = \partial E \cap \partial C$ and assume that μ contains l vertices of M other than u, v and w. If one of them has valency at least 4 in M then C contains a reducing chain of the second kind. See Fig. 12(c). So let us assume that all of them have valency 3 in M. If $l \geqq 2$, M contains a reducing chain of the fourth kind. See Fig. 12(d).

Let $l = 1$. See Fig. 14. If $\partial D_i \cap \partial M' \subseteq \partial E$ then M contains an exceptional reducing chain. Let F be the neighbour of D_j in C_j. If $\partial F \cap \partial M' \subseteq \partial E$ then M contains a reducing chain of the third or fourth kind. So assume that $\partial D_i \cap \partial M' \not\subseteq \partial E$ and $\partial F \cap \partial M' \not\subseteq E$. (See Fig. 12(a).) If $E \in \mathrm{Cor}_2(M')$ then by $(*)$ at least one of the vertices u, v has valency 3 in M and if $E \in \mathrm{Cor}_3(M')$ then by $(**)$ both u and v have valency 3 in M. But both possibilities contradict W(4).

Let $l = 0$. If $\partial D_i \cap \partial M' \subseteq \partial E$ and $\partial D_j \cap \partial M' \subseteq \partial E$ then M has a reducing chain of the first kind or the second kind or is of exceptional kind. (See Fig. 13(a).)

If $\partial D_i \cap \partial M' \not\subseteq \partial E$ and $\partial D_j \cap \partial M' \not\subseteq \partial E$, we get a contradiction to W(4), in view of $(*)$ and $(**)$.

Let $\partial D_i \cap \partial M' \subseteq \partial E$ and let $\partial D_j \cap \partial M' \not\subseteq E$. If u has valency $\geqq 4$ in M' then by $(*)$ v has valency 3 in M' and we get a contradiction to W(4).

Therefore u has valency 3 in M', so we obtain the situation in part (c) of the lemma.

Similarly, if $\partial D_i \cap \partial M' \not\subseteq \partial E$, $\partial D_j \cap \partial M' \subseteq \partial E$, we obtain part (c) of the lemma. See Fig. 13(b).

The lemma is proved.

Fig. 13.

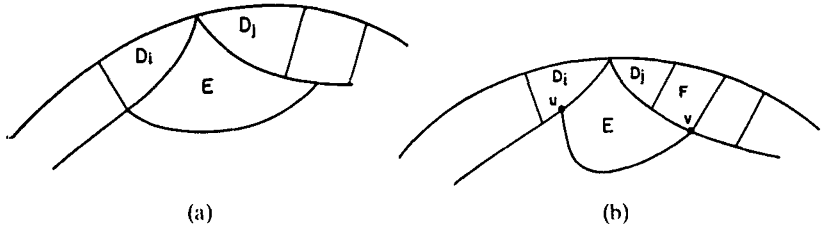


Fig. 14.

LEMMA 4. *Let M be a simply connected map with a connected interior which contains more than one region. If M satisfies the condition* W(4) *and* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *hold then M has a boundary region E with* $i_M(E) = 1$.

PROOF. It follows from [1, Theorem A] that M has a maximal boundary chain C. Let M' be the submap obtained by deleting C from M. By $\mathcal{H}_2$(a) either M' consists of a simple region or M' has a reducing chain S. In this case by Lemma 2, $\partial S \cap \partial M' \subseteq \partial C$. Assume first that M' contains more than one region and M has no region with $i_M(D) = 1$. We claim that S is a simply connected exceptional reducing chain. First, by Definition 3 every extremal region of S either satisfies (*) or satisfies (**) of Lemma 3, hence by Lemma 3 satisfies part (c) of Lemma 3. This immediately rules out reducing chains of the first, fourth, fifth, and exceptional chains with $r = 2$. Also, S is not of the sixth kind, by Lemma 2. On the other hand, the arguments given in Lemma 3 easily show that if S would be of the second kind then it would produce an exceptional reducing chain, if S would be of the third kind then it would produce a reducing chain of the third kind and if S would be exceptional then it would produce a reducing chain of the third kind of M. This however violates hypotheses $\mathcal{H}_1$ or $\mathcal{H}_2$. Thus S necessarily is exceptional with $r \geqq 3$. If S is annular then the two extremal regions of S form a reducing chain of the third kind, again violating $\mathcal{H}_1$ or $\mathcal{H}_2$. Hence S is a simply connected exceptional reducing chain.

Let Σ be the minimal connected submap of C such that $\partial\Sigma \supseteq \partial C \cap \partial S$. (See Fig. 15.)

We claim that Σ contains an exceptional reducing chain. Indeed, let x and y be the endpoints of $\partial S \cap \partial C$. By Definition 3(g) $d_{M'}(x) = d_{M'}(y) = 3$, while by Lemma 3(c) $d_M(x) = d_M(y) = 4$. Consequently

(i) $\partial\Sigma \cap \partial M' = \partial S \cap \partial M'$.

Furthermore

(ii) every component of the interior of Σ consists of a single region and

(iii) every region of S has a nontrivial common boundary with exactly two regions (hence components) of Σ.

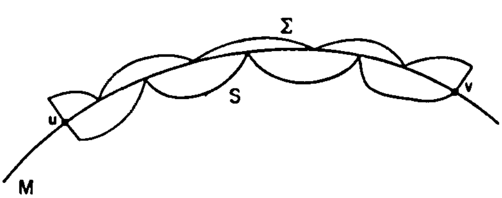
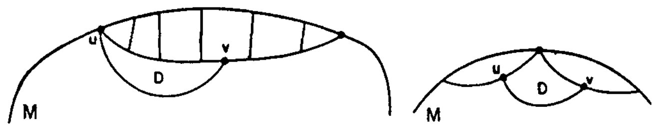


Fig. 15.



(a) (b)

Fig. 16.

To see (ii) and (iii) we show first that if D is a non-extremal region of S then the endpoints u and v of $\partial D \cap \partial \Sigma$ cannot belong to the same component of the interior of Σ. (See Fig. 16.)

Assume not and let $\mu = \partial D \cap \partial \Sigma$. (See Fig. 16(a).) By $\mathcal{K}_1$ and $\mathcal{K}_2$, we may assume that all the vertices of μ, except u and v, have valency 3. Consequently, by the W(4) condition μ has at least 3 vertices, u and v excluded. But then Σ contains a reducing chain of the fifth kind, contradiction. Thus u and v must be on different components of the interior of Σ. (See Fig. 16(b).)

Assume now that (ii) is false. First we claim that every component of the interior of Σ which contains more than one region, in fact contains exactly two regions. It is clear that either u or v must belong to an extremal region of some component of the interior of Σ, for otherwise Σ contains an exceptional reducing chain of the second kind (see Fig. 14(a)). We may assume that u and v belong to adjacent components C_i and C_{i+1} of the interior of Σ. (The arguments of the proof of Case 2 in the proof of Lemma 3 apply here.) Let D_i and D_{i+1} be the extremal regions of C_i and C_{i+1} respectively such that $\partial D_i \cap \partial D_{i+1} \neq \varnothing$ and assume that $u \in \partial D_i$. If $v \notin \partial D_{i+1}$ and F is the neighbour of D_{i+1} then v necessarily belongs to F (see Fig. 14(b)) for otherwise the submap containing D_{i+1} and F constitutes a reducing chain of the fourth kind. Since the two endpoints of the common edge of a region of S with $\partial \Sigma$ cannot be on the same component of Σ, this implies that every component C_j of S contains at most three regions for otherwise we would have a reducing chain of the fifth kind. But if C_j contains 3 regions then C_j constitutes a reducing chain of the sixth kind. Thus every component of the interior of Σ contains at most two regions.

Let us label the components of Σ as follows:

Let $\nu = \partial S \cap \partial \Sigma$. Then the component which contains $0(\mu)$ and is to the right of $t(\mu)$ has subscript 1, the component following it has subscript 2, and so on.

Let now $C_{i_1}, \ldots, C_{i_t}$ be all the components of the interior of Σ which contain two regions. Assume that the components are labelled in such a way that if $i_j < i_k$

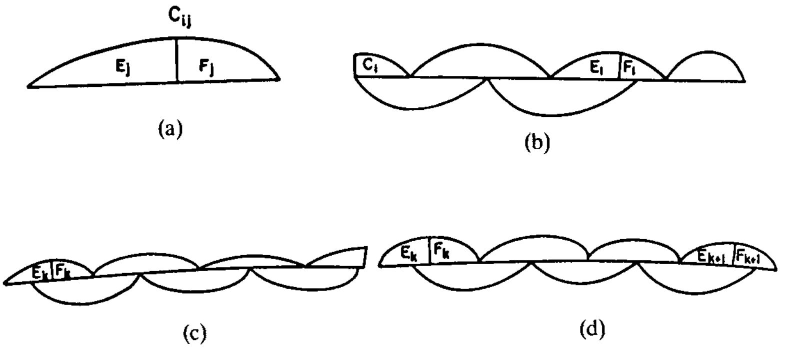


Fig. 17.

then C_{i_j} is to the left of C_{i_k}. Denote by E_j and F_j the regions of C_{i_j}, such that E_j is to the left of F_j (see Fig. 17(a)).

Let us consider C_{i_1}. It consists of E_1 and F_1. It follows easily from the W(4) condition and the assumptions $\mathcal{K}_1$ and $\mathcal{K}_2$ that C_{i_1} has exactly 2 neighbours in S. Consequently either E_1 or F_1 has exactly one neighbour in S. In the first case the subchain of Σ beginning with C_1 and ending with E_1 is an exceptional reducing chain. (See Fig. 17(b).) In the second case let k be the highest index such that F_k has exactly one neighbour in S. If $k = t$ then the tail subchain of Σ beginning with F_1 is an exceptional reducing chain. (See Fig. 17(c).) On the other hand, if $k < t$ then the subchain of Σ beginning with F_k and ending with E_{k+1} is again an exceptional reducing chain. (See Fig. 17 (d).) This contradiction proves that (ii) holds. Due to (ii), if (iii) is false then Σ has a region D with $i_M(D) = 1$, contradicting our assumption. Thus (iii) holds. But then Σ is an exceptional reducing chain, violating $\mathcal{K}_1$ or $\mathcal{K}_2$. Consequently M' contains at most one region. However in this case the lemma is immediate. This completes the proof of Lemma 4.

Proof of Theorem 1. If M has only one region the statement is vacuous. So assume M has more than one region and prove the theorem by induction on $V(M)$. Thus assume $\mathcal{K}_1$ and $\mathcal{K}_2$. Then by Lemma 4 either M has a reducing chain, in which case $\mathcal{K}_2$ is violated, or M has a region E with $i_M(E) = 1$, in which case $\mathcal{K}_1$ is violated. This completes the proof of the theorem.

Now let us eliminate the case when a boundary region E with $i_M(E) = 1$ exists. This we shall do through the next Lemma.

Lemma 5. *Let M be a simply connected map with a connected interior.*

Assume that M satisfies the condition W(4). *Let M' be the submap of M obtained by deleting all the regions E with $i_M(E)=1$ from M. If $\mathcal{K}_2$ holds then*

(a) *M' is connected and*

(b) *M' has no boundary regions F with $i_{M'}(F)=1$.*

PROOF. Follows easily from the fact that all the regions F deleted from M satisfy $i_M(F)=1$.

(b) Let F be a boundary region of M' with $i_{M'}(F)=1$. If $i_M(F)=k$ then F has $k-1$ neighbours E_j with $i_M(E_j)=1$ which are not in M'. If $\partial E_j \cap \partial M$ has an endpoint with valency 3 then $\{E_j\}$ is a reducing chain of the first kind, contradicting $\mathcal{K}_2$. Thus the endpoints of $\partial E_j \cap \partial M$ have valency at least 4, for all $j=1,\dots,k-1$. On the other hand, since every E_j has exactly one neighbour in M, namely F, and F has only one neighbour in M', no common vertex of ∂E_j and ∂E_{j+1} may have valency greater than 4. (See Fig. 18.)

Therefore, if $k-1 \geqq 2$ then $\{E_1, E_2\}$ constitutes an exceptional reducing chain of the first kind, violating $\mathcal{K}_2$. Thus $k-1 \leqq 1$, i.e., $k \leqq 2$ and $i_M(F) \leqq 2$. By the W(4) condition this implies that $\partial F \cap \partial M$ contains an edge. But then $\partial E_1 \cap \partial M$ has an endpoint with valency 3, contradicting $\mathcal{K}_2$ again. Consequently M' has no boundary region F with $i_{M'}(F)=1$, as required. This completes the proof of the Lemma.

PROOF OF THEOREM A. If M has only one region, we are done. So assume M has more than one region. If M has a reducing chain then the theorem follows by Remark 1. So assume $\mathcal{K}_2$(b). Then by Lemma 4 we may assume that M has a region D with $i_M(D)=1$. Let M' be the map obtained by deleting all the regions D of M with $i_M(D)=1$. Then M' is connected by Lemma 5 and $M' \neq M$. Consequently, by Lemma 3 and Lemma 4, if M' contains more than one region, which we certainly may assume, then M' has a reducing chain S. We claim that if $\partial S \cap \partial D \neq \varnothing$ for some D with $i_M(D)=1$ then $\partial D \cap \partial S$ contains only one vertex. Indeed, if $\partial D \cap \partial E$ contains an edge for a (hence a unique) region E of S then $d_M(E)=d_{M'}(E)+1 \leqq 5$ by Definition 3 and equality holds only if S is of the sixth kind in which case E has two vertices at least with valency 3, which

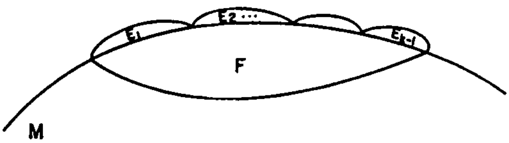


Fig. 18.

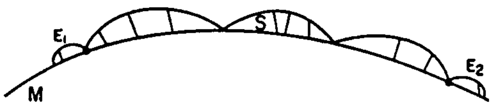


Fig. 19.

contradicts W(4). If $\partial E \cap \partial M$ does not contain an edge then this violates W(4). Also if $d_M(E) = 4$ then E has a vertex with valency 3 by Definition 3, again violating the condition W(4). Consequently $\partial D \cap \partial E$ cannot contain an edge, unless ∂E has an edge on ∂M. But in this case ∂D has a vertex with valency 3, contradicting $\mathcal{K}_2$. Thus S is a boundary chain of M, but not necessarily a reducing chain, because the valency of the two extremal points may increase. See Fig. 19. Let M'' be the map obtained by deleting S from M'. Then

(*) M'' is obtained from M by deleting at most $\beta(M)$ boundary regions from M.

On the other hand, certainly $\beta(M') = \beta(M)$, while $\beta(M'') < \beta(M')$ by Theorem 1. Thus

(**) $\beta(M'') < \beta(M)$.

Now (*) and (**) imply the theorem by Remark 1.

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